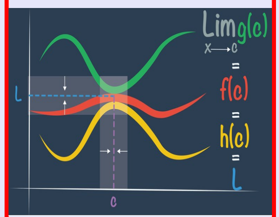
Calculus I

Lecture 13



Feb 19-8:47 AM

Given $f(x)=\dfrac{x^2}{\sqrt{x+1}}$, $f'(x)=\dfrac{x(3x+4)}{2(x+1)^{3/2}}$

$f''(x)=\dfrac{3x^2+8x+8}{4(x+1)^{5/2}}$

1) Domain $x+1\neq 0$ $x\neq -1$

$x+1>0$ $x>-1$ → $(-1,\infty)$

2) All intercepts

Y-Int $(0,0)$

X-Int $(0,0)$ Twice

3) Asymptotes

V.A. $x=-1$

$\lim_{x\to\infty} \dfrac{x^2}{\sqrt{x+1}}=\infty$

4) C.N.

$f'(x)=0$ or undefined

$x(3x+4)=0$ $x+1=0$

$x=0$ $\cancel{x=-\frac{4}{3}}$ $x=-1$

5) P.I.P. locations

$f''(x)=0$ or undefined

$3x^2+8x+8=0$ $x+1=0$

$x=-1$

$x=\dfrac{-b\pm\sqrt{b^2-4ac}}{2a}$

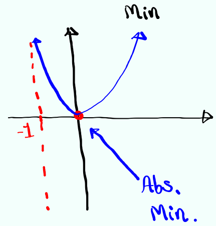
$=\dfrac{-8\pm\sqrt{64-96}}{6}$

$=\dfrac{-8\pm\sqrt{-}}{6}$ No Real Solution

6) Sign Chart

x	-1		0		∞
$f'(x)$		$-$	$\bullet$	$+$	
$f''(x)$		$+$		$+$	
$f(x)$			Min		

Range $[0,\infty)$



Jan 27-8:03 AM

Given $x^2 + y^2 + z^2 = 29$

$x = x(t)$, $\frac{dx}{dt} = 3$

$y = y(t)$, $\frac{dy}{dt} = 5$

$z = z(t)$,

$2x\frac{dx}{dt} + 2y\frac{dy}{dt} + 2z\frac{dz}{dt} = 0$

$2 \cdot 3 + 3 \cdot 5 + 4\frac{dz}{dt} = 0$

$6 + 15 + 4\frac{dz}{dt} = 0$

Find $\frac{dz}{dt}$ at $(x, y, z) = (2, 3, 4)$

$\boxed{\frac{dz}{dt} = \frac{-21}{4}}$

As x & y increase, z will be decreasing at (2,3,4)

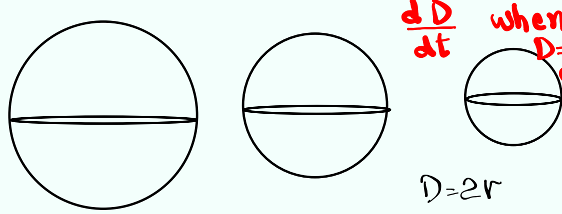
Jan 27-8:25 AM

Snowball is melting such that its surface area decreases at 1 cm²/min.

$\frac{dS}{dt} = -1\,cm^2/min$

Find the rate at which the diameter decreases when diameter is 10 cm.

$\frac{dD}{dt}$ when D=10 cm



Formula for Surface area?

$V = \frac{4\pi r^3}{3}$

$S = 4\pi r^2$

$D = 2r$

$\frac{D}{2} = r$

$S = 4\pi\left(\frac{D}{2}\right)^2$

$S = \cancel{4}\pi \frac{D^2}{\cancel{4}}$

$S = \pi D^2$

$\frac{dS}{dt} = \pi \cdot 2D \frac{dD}{dt}$

$-1 = 2\pi(10)\frac{dD}{dt}$

$\boxed{\frac{dD}{dt} = \frac{-1}{20\pi}\ cm/min.}$

Jan 27-8:31 AM

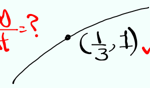
A particle is moving along the curve

$y = 2\sin\left(\frac{\pi x}{2}\right)$. $\qquad \frac{dx}{dt} = \sqrt{10}$ cm/s

x-coordinate increases at $\sqrt{10}$ cm/s.

How fast the $\boxed{\text{y-coordinate change}}$ at $\left(\frac{1}{3}, 1\right)$?

$\frac{dy}{dt} = ?$



$y = 2\sin\left(\frac{\pi}{2}\cdot\frac{1}{3}\right)$

$= 2\sin\frac{\pi}{6}$

$= 2\cdot\sin 30^\circ$

$= 2\cdot\frac{1}{2} = 1$

$y = 2\sin\left(\frac{\pi x}{2}\right)$

$\frac{dy}{dt} = 2\cdot\cos\left(\frac{\pi x}{2}\right)\cdot\frac{d}{dt}\left[\frac{\pi x}{2}\right]$

$\frac{dy}{dt} = 2\cos\frac{\pi x}{2}\cdot\frac{\pi}{2}\cdot\frac{dx}{dt}$

$= \cos\left(\frac{\pi}{2}\cdot\frac{1}{3}\right)\cdot\pi\cdot\sqrt{10}$

$= \cos\frac{\pi}{6}\cdot\pi\sqrt{10} = \frac{\sqrt{3}}{2}\cdot\pi\sqrt{10}$

$= \boxed{\frac{\pi\sqrt{30}}{2}\text{ cm/s}}$

y-coordinate is increasing.

Jan 27-8:40 AM

Use quadratic approximation to estimate

$\sqrt[3]{1001}$

1) Common Sense

$\sqrt[3]{1001} \approx \sqrt[3]{1000} = 10$

2) Using Calculator

$\sqrt[3]{1001} \approx 10.00333222$

3) $f(x) = \sqrt[3]{x}$, $a = 1000$

$f(a) = \sqrt[3]{1000} = 10$

$f(x) = x^{1/3}$

$f'(x) = \frac{1}{3}x^{-2/3} = \frac{1}{3(\sqrt[3]{x})^2}$

$f'(a) = f'(1000) = \frac{1}{3(\sqrt[3]{1000})^2} = \frac{1}{300}$

$f''(x) = \frac{1}{3}\cdot\frac{-2}{3}x^{-5/3} = \frac{-2}{9(\sqrt[3]{x})^5}$

$f''(a) = f''(1000) = \frac{-2}{9(\sqrt[3]{1000})^5} = \frac{-2}{9\cdot 10^5}$

$= \frac{-2}{9\cdot 100000} = \frac{-1}{450000}$

$f(x) \approx f(a) + f'(a)(x-a) + \frac{f''(a)}{2}(x-a)^2$

$\sqrt[3]{x} \approx 10 + \frac{1}{300}(x-1000) - \frac{1}{900000}(x-1000)^2$

when $x = 1001$

$\sqrt[3]{1001} \approx 10 + \frac{1}{300} - \frac{1}{900000}$

$\approx \boxed{10.00333222}$

Jan 27-8:49 AM

$f(x) = x^4 - 4x^3 = x^3(x-4)$ $f(3) = 3^4 - 4(3)^3 = 81 - 108 = -27$

$f(2) = -16$

1) Domain
$f(x)$ is polynomial.
$(-\infty, \infty)$

2) Asymptotes
None

3) Intercepts
Y-Int $(0,0)$
x-Int $(0,0)$ 3-times
$(4,0)$ 1-time

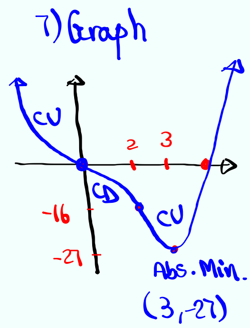
4) Critical points
$f'(x) = 4x^3 - 12x^2$
$= 4x^2(x-3)$
$(0,0)$, $(3,-27)$

5) Possible inflection Points
$f''(x) = 12x^2 - 24x$
$= 12x(x-2)$
$(0,0)$
$(2,-16)$

6) Sign Chart

x	$-\infty$ to 0	0 to 2	2 to 3	3 to ∞
f'	−	−	−	+
f''	+	−	+	+
f	↘	↘	↘	↗

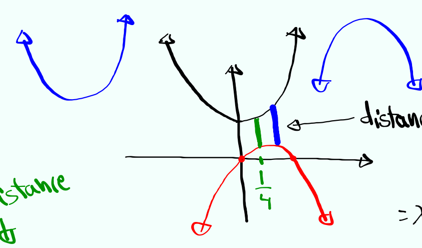
7) Graph



8) Range
$[-27, \infty)$

Jan 27-9:02 AM

Find the minimum $\boxed{\text{Vertical distance}}$ between $\boxed{y = x^2 + 1}$ and $\boxed{y = x - x^2}$?



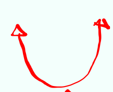
distance = Top Y − Bottom Y
$= x^2 + 1 - (x - x^2)$
$= 2x^2 - x + 1$
Minimize

distance

$f(x) = 2x^2 - x + 1$

$f'(x) = 4x - 1$

$f''(x) = 4 > 0$



$f'(x) = 0$
$x = \frac{1}{4}$

$f(\frac{1}{4})$

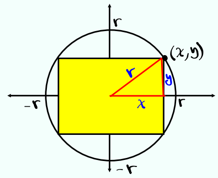
$f(\frac{1}{4}) = 2(\frac{1}{4})^2 - \frac{1}{4} + 1$

$= 2 \cdot \frac{1}{16} - \frac{1}{4} + 1 = \frac{1}{8} - \frac{1}{4} + 1$

$= \frac{1 - 2 + 8}{8} = \boxed{\frac{7}{8}}$

Jan 27-9:24 AM

Find the dimensions of the rectangle with $\boxed{\text{largest area}}$ can can be inscribed in a circle with radius r.



Dimensions $2x$ by $2y$

Area $= 2x \cdot 2y$

$= 4xy$

Maximize

$x^2 + y^2 = r^2$

$y^2 = r^2 - x^2$

$y = \sqrt{r^2 - x^2}$

$y = \sqrt{r^2 - \frac{r^2}{2}}$

$= \frac{r}{\sqrt{2}}$

Area

$f(x) = 4x\sqrt{r^2 - x^2}$

$f(x) = 4\,x\,(r^2 - x^2)^{1/2}$

$f'(x) = 4\left[1(r^2 - x^2)^{1/2} + x \cdot \frac{1}{2}(r^2 - x^2)^{-1/2} \cdot (-2x)\right]$

$= 4\left[\sqrt{r^2 - x^2} - \frac{x^2}{\sqrt{r^2 - x^2}}\right] = 4 \cdot \frac{r^2 - x^2 - x^2}{\sqrt{r^2 - x^2}}$

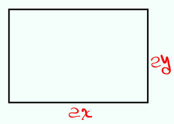
$f'(x) = \frac{4(r^2 - 2x^2)}{\sqrt{r^2 - x^2}}$ $\qquad f'(x) = 0$ $\qquad r^2 - 2x^2 = 0$

$x^2 = \frac{r^2}{2}$

$x = \frac{r}{\sqrt{2}}$

$+$ Max. $-$

$\frac{r}{\sqrt{2}}$



Dimensions

$\frac{2r}{\sqrt{2}}$ by $\frac{2r}{\sqrt{2}}$

$r\sqrt{2}$ by $r\sqrt{2}$

Jan 27-9:32 AM

Given $f'(x) = \boxed{5x^4} - \boxed{3x^2} + \boxed{4}$, $f(-1) = 2$

Find $f(x)$.

$f(x) = x^5 - x^3 + 4x + C$

$f(-1) = (-1)^5 - (-1)^3 + 4(-1) + C = 2$

$\cancel{-1} \quad \cancel{+1} \quad -4 \quad + C = 2$

$\boxed{C = 6}$

$\boxed{f(x) = x^5 - x^3 + 4x + 6}$

Jan 27-9:47 AM

Recall

$$\lim_{x\to\infty} \frac{1}{x} = 0 \qquad \lim_{x\to\infty} \frac{1}{x^n} = 0 \text{ when } n>0$$

Evaluate $\lim_{x\to\infty} \frac{2x-5}{5x-2} = \frac{\infty}{\infty}$ I.F.

Divide by x to largest power.

$$\lim_{x\to\infty} \frac{\frac{2x-5}{x}}{\frac{5x-2}{x}} = \lim_{x\to\infty} \frac{\frac{2x}{x} - \frac{5}{x}}{\frac{5x}{x} - \frac{2}{x}}$$

$$= \lim_{x\to\infty} \frac{2 - \cancel{\frac{5}{x}} \to 0}{5 - \cancel{\frac{2}{x}} \to 0}$$

$$= \boxed{\frac{2}{5}}$$

Jan 27-10:13 AM

Evaluate

1) $\lim_{x\to\infty} \frac{3x^2-x+4}{2x^2+5x-8}$ $\frac{\infty}{\infty}$ I.F.

Divide by x^2

$$= \lim_{x\to\infty} \frac{\frac{3x^2}{x^2} - \frac{x}{x^2} + \frac{4}{x^2}}{\frac{2x^2}{x^2} + \frac{5x}{x^2} - \frac{8}{x^2}} = \lim_{x\to\infty} \frac{3 - \cancel{\frac{1}{x}} \to 0 + \cancel{\frac{4}{x^2}} \to 0}{2 + \cancel{\frac{5}{x}} \to 0 - \cancel{\frac{8}{x^2}} \to 0}$$

$$= \boxed{\frac{3}{2}}$$

2) $\lim_{x\to\infty} \frac{1-x^2}{x^3-x+1} = \frac{-\infty}{\infty}$ I.F.

Divide by x^3

$$= \lim_{x\to\infty} \frac{\frac{1}{x^3} - \frac{x^2}{x^3}}{\frac{x^3}{x^3} - \frac{x}{x^3} + \frac{1}{x^3}} = \lim_{x\to\infty} \frac{\cancel{\frac{1}{x^3}} \to 0 - \cancel{\frac{1}{x}} \to 0}{1 - \cancel{\frac{1}{x^2}} \to 0 + \cancel{\frac{1}{x^3}} \to 0}$$

$$= \frac{0}{1} = \boxed{0}$$

Jan 27-10:18 AM

Evaluate $\lim\limits_{x\to\infty} \dfrac{x^2}{\sqrt{25x^4+1}}$ $\quad \dfrac{\infty}{\infty}$ I.F.

$$\frac{x^2}{\sqrt{25x^4+1}} \approx \frac{x^2}{\sqrt{25x^4}} = \frac{x^2}{5x^2} = \frac{1}{5}$$

Divide by x^2

$$= \lim_{x\to\infty} \frac{\frac{x^2}{x^2}}{\frac{\sqrt{25x^4+1}}{x^2}}$$

$$= \lim_{x\to\infty} \frac{1}{\sqrt{\frac{25x^4+1}{x^4}}} = \lim_{x\to\infty} \frac{1}{\sqrt{\frac{25x^4}{x^4}+\frac{1}{x^4}}}$$

$$= \lim_{x\to\infty} \frac{1}{\sqrt{25+\cancel{\frac{1}{x^4}}\to 0}} = \frac{1}{\sqrt{25}} = \boxed{\frac{1}{5}}$$

Jan 27-10:25 AM

Evaluate $\lim\limits_{x\to\infty} \dfrac{\sqrt{9x^6-x}}{x^3+1} = \dfrac{\infty}{\infty}$ I.F.

$$\frac{\sqrt{9x^6-x}}{x^3+1} \approx \frac{\sqrt{9x^6}}{x^3} = \frac{3x^3}{x^3} \approx \boxed{3}$$

$$(x^m)^n = x^{mn}$$

Divide by x^3

$$= \lim_{x\to\infty} \frac{\frac{\sqrt{9x^6-x}}{x^3}}{\frac{x^3+1}{x^3}}$$

$$= \lim_{x\to\infty} \frac{\sqrt{\frac{9x^6-x}{x^6}}}{1+\frac{1}{x^3}} = \lim_{x\to\infty} \frac{\sqrt{9-\cancel{\frac{1}{x^5}}\to 0}}{1+\cancel{\frac{1}{x^3}}\to 0} = \frac{\sqrt{9}}{1} = \boxed{3}$$

Jan 27-10:30 AM

Evaluate $\lim_{x \to -\infty} \frac{\sqrt{x^2+1}}{2x-1}$ $\frac{\infty}{-\infty}$ I.F.

$\frac{\sqrt{x^2+1}}{2x-1} \approx \frac{\sqrt{x^2}}{2x} = \frac{x}{2x} = \frac{1}{2}$

when $x \to \infty$, $x = \sqrt{x^2}$

when $x \to -\infty$ $x = -\sqrt{x^2}$

Divide by x

$$\lim_{x \to -\infty} \frac{\frac{\sqrt{x^2+1}}{x}}{\frac{2x-1}{x}} = \lim_{x \to -\infty} \frac{-\sqrt{\frac{x^2+1}{x^2}}}{2-\frac{1}{x}} = \lim_{x \to -\infty} \frac{-\sqrt{1+\cancel{\frac{1}{x^2}}^{0}}}{2-\cancel{\frac{1}{x}}^{0}} = \boxed{-1/2}$$

Let $x = -1000$

$$\frac{\sqrt{(-1000)^2+1}}{2(-1000)-1} = -.4997503748 \approx \frac{-1}{2}$$

$\sqrt{x^2} \geq 0$ $\quad \overset{+}{\sqrt{x^2}} = \overset{+}{-x}$

what if $x < 0$

$\sqrt{(-2)^2} = |-2| = 2$

$\sqrt{2^2} = |2| = 2$

$\sqrt{x^2} = |x|$

$x \to -\infty$

$$|x| = \begin{cases} \boxed{-x} & \text{if } x < 0 \\ x & \text{if } x \geq 0 \end{cases}$$

Jan 27-10:36 AM

Evaluate $\lim_{x \to \infty} x \sin \frac{1}{x} = \infty \cdot \boxed{\sin 0}$

$= \infty \cdot 0$

I.F.

$$= \lim_{x \to \infty} x \sin \frac{1}{x} = \lim_{h \to 0} \frac{1}{h} \cdot \sin h = \lim_{h \to 0} \frac{\sin h}{h} = \boxed{1}$$

Let $h = \frac{1}{x}$

$x \to \infty$, $h \to 0$ $\to xh = 1$ $\quad x = \frac{1}{h}$

Let $x = 1000$

$$1000 \cdot \sin \frac{1}{1000} \approx .9999998333$$

$$\approx 1$$

Jan 27-10:51 AM

Evaluate $\lim_{x\to\infty}\left(\sqrt{x^2+4x}-x\right) = \infty - \infty$

I.F.

use conjugate to rationalize.

$\frac{0}{0}, \frac{\infty}{\infty}, \infty - \infty, \infty \cdot 0$

$$= \lim_{x\to\infty} \frac{(\overset{A}{\sqrt{x^2+4x}} \overset{-B}{-x})(\overset{A}{\sqrt{x^2+4x}} \overset{+B}{+x})}{\sqrt{x^2+4x}+x}$$

$$= \lim_{x\to\infty} \frac{\overset{A^2}{x^2+4x} \overset{-B^2}{-x^2}}{\sqrt{x^2+4x}+x} = \lim_{x\to\infty} \frac{4x}{\sqrt{x^2+4x}+x} \quad \text{I.F. } \frac{\infty}{\infty}$$

$$= \lim_{x\to\infty} \frac{\frac{4x}{x}}{\frac{\sqrt{x^2+4x}}{x}+\frac{x}{x}} = \lim_{x\to\infty} \frac{4}{\sqrt{\frac{x^2+4x}{x^2}}+1}$$

$$= \lim_{x\to\infty} \frac{4}{\sqrt{1+\cancel{\frac{4}{x}}^{\to 0}}\;+1} = \frac{4}{\sqrt{1}+1} = \frac{4}{2} = \boxed{2}$$

Jan 26-11:56 AM

Evaluate $\lim_{x\to\infty}\left(\sqrt{9x^2-5x}-3x\right) = \infty - \infty$ I.F.

$$= \lim_{x\to\infty} \frac{(\sqrt{9x^2-5x}-3x)(\sqrt{9x^2-5x}+3x)}{\sqrt{9x^2-5x}+3x}$$

$$= \lim_{x\to\infty} \frac{(\sqrt{9x^2-5x})^2-(3x)^2}{\sqrt{9x^2-5x}+3x} = \lim_{x\to\infty} \frac{-5x}{\sqrt{9x^2-5x}+3x} = \frac{-\infty}{\infty} \text{ I.F.}$$

Divide by x.

$$= \lim_{x\to\infty} \frac{\frac{-5x}{x}}{\sqrt{\frac{9x^2}{x^2}-\frac{5x}{x^2}}+\frac{3x}{x}} = \lim_{x\to\infty} \frac{-5}{\sqrt{9-\cancel{\frac{5}{x}}^{\to 0}}+3} = \boxed{\frac{-5}{6}}$$

$= -.8\overline{3}$

Let $x = 1000$

$$\sqrt{9(1000)^2-5(1000)} - 3(1000) \approx -.8334491062$$

Jan 27-11:04 AM

$f(x) = \dfrac{x^3}{x^2+1}$

1) Domain $x^2+1=0$

$(-\infty, \infty)$ $\quad x^2 = -1$

No real Sol.

2) V.A.

None

3) Intercepts

y-Int $(0,0)$

x-Int $(0,0)$

3-times

4) Show $f(x)$ is an odd function

$f(-x) = \dfrac{(-x)^3}{(-x)^2+1} = \dfrac{-x^3}{x^2+1} = -\dfrac{x^3}{x^2+1}$

$= -f(x)$

Symmetric w/t origin

5) $\lim\limits_{x\to\infty} f(x)$

$= \lim\limits_{x\to\infty} \dfrac{x^3}{x^2+1} = \infty$

6) $\lim\limits_{x\to-\infty} f(x) = -\infty$

Jan 27-11:14 AM

7) $f'(x) = \dfrac{x^2(x^2+3)}{(x^2+1)^2}$

C.N.

$f'(x)=0$, undefined

$x=0$

8) $f''(x) = \dfrac{2x(3-x^2)}{(x^2+1)^3}$

location for P.I.P.

$f''(x)=0$ or undefined

$x=0, \quad x=\pm\sqrt{3}$

9) Sign chart

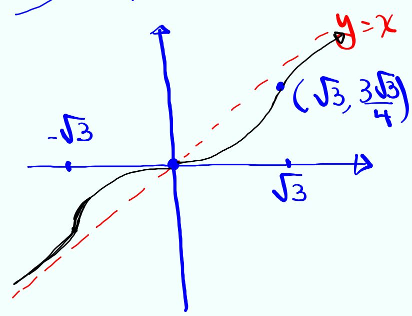
x	$-\infty$	$-\sqrt{3}$		0		$\sqrt{3}$		∞
f'	$+$		$+$		$+$		$+$	
f''	$+$		$-$		$+$		$-$	
$f(x)$	↗		↗		↗		↗	

$f(x) = \dfrac{x^3}{x^2+1} \approx \dfrac{x^3}{x^2} \approx x$

Slant Asymptote

$y = x$

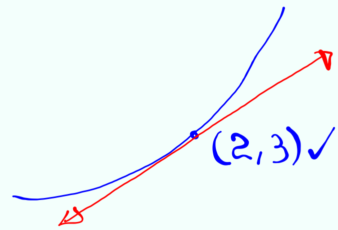
$f(\sqrt{3}) = \dfrac{(\sqrt{3})^3}{(\sqrt{3})^2+1} = \dfrac{3\sqrt{3}}{4}$



Jan 27-11:22 AM

Find eqn of the tan. line to the graph of

$4x^2+y^2=25$ at $(2,3)$.



$$8x+2y\frac{dy}{dx}=0$$

$$\frac{dy}{dx}=\frac{-4x}{y}$$

$$m=\frac{dy}{dx}(2,3)=\frac{-8}{3}$$

$$y-3=\frac{-8}{3}(x-2)$$

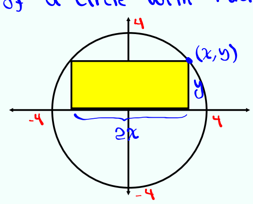
$$3y-9=-8(x-2)$$

$$3y-9=-8x+16$$

$$\boxed{3y+8x=25}$$

Jan 27-11:38 AM

Find the area of the largest rectangle that can be incribed in the top half of a circle with radius 4.



$x^2+y^2=4^2$

$y^2=16-x^2$

Top half

$y=\sqrt{16-x^2}$

$A=2xy$

$y=\sqrt{16-8}$
$=\sqrt{8}$
$=2\sqrt{2}$

$A(x)=2x\sqrt{16-x^2}$

$=2\sqrt{x^2(16-x^2)}=2(16x^2-x^4)^{1/2}$

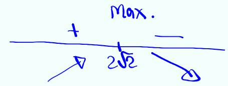
$A'(x)=2\cdot\frac{1}{2}(16x^2-x^4)^{-1/2}\cdot(32x-4x^3)$

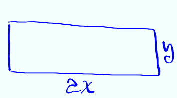
$A'(x)=\frac{32x-4x^3}{\sqrt{16x^2-x^4}}=\frac{x(32-4x^2)}{x\sqrt{16-x^2}}=\frac{4(8-x^2)}{\sqrt{16-x^2}}$

$A'(x)=0$ $\quad 8-x^2=0$ $\quad x^2=8$ $\quad x=\sqrt{8}$

Max.

$x=2\sqrt{2}$





$A=2xy$

$=2\cdot 2\sqrt{2}\cdot 2\sqrt{2}$

$=2\cdot 2\cdot\sqrt{4}\cdot 2$

$=\boxed{16}$

Jan 27-11:49 AM